Coherent Lagrangian vortices

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Joint work with:

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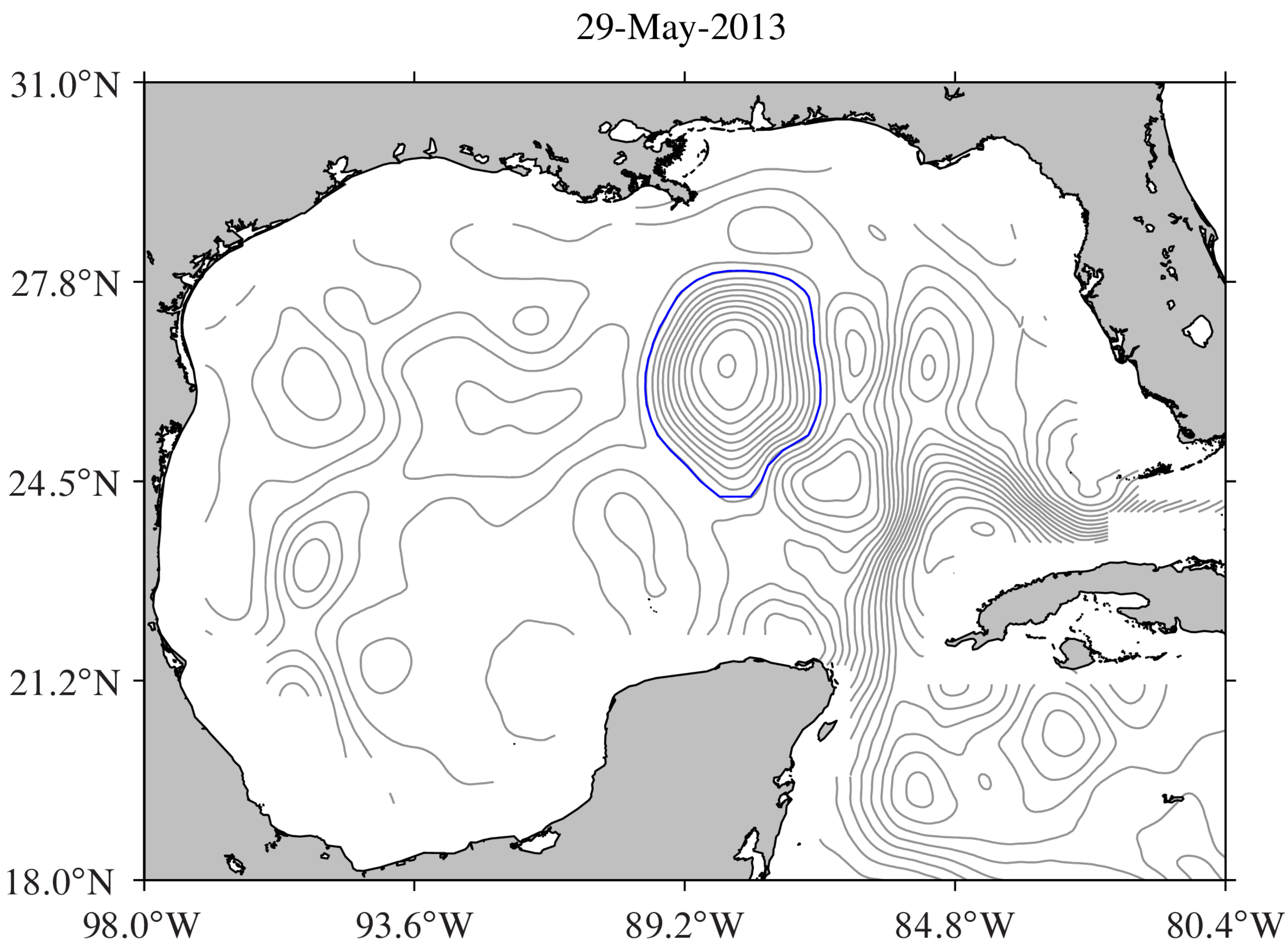
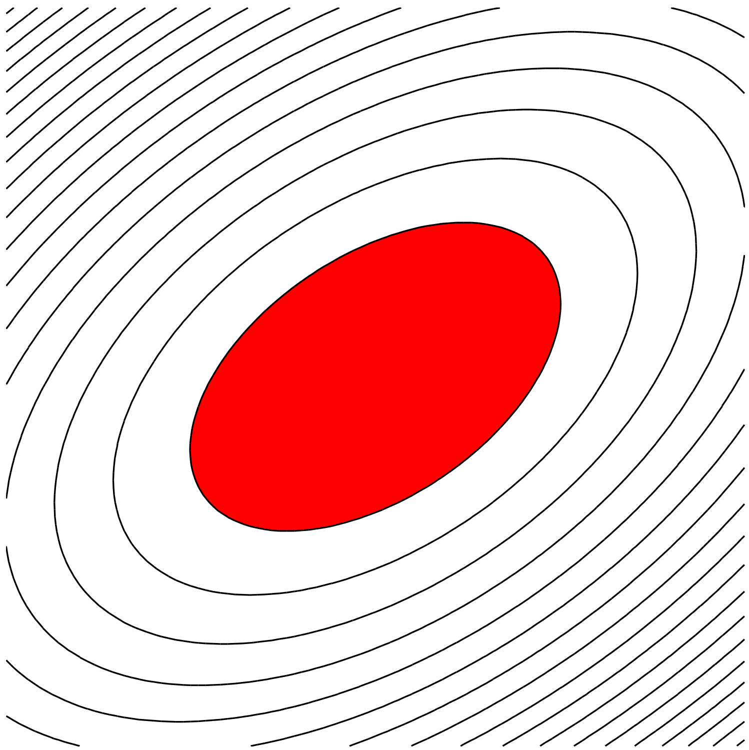
Evolution of passive tracers in the Gulf of Mexico

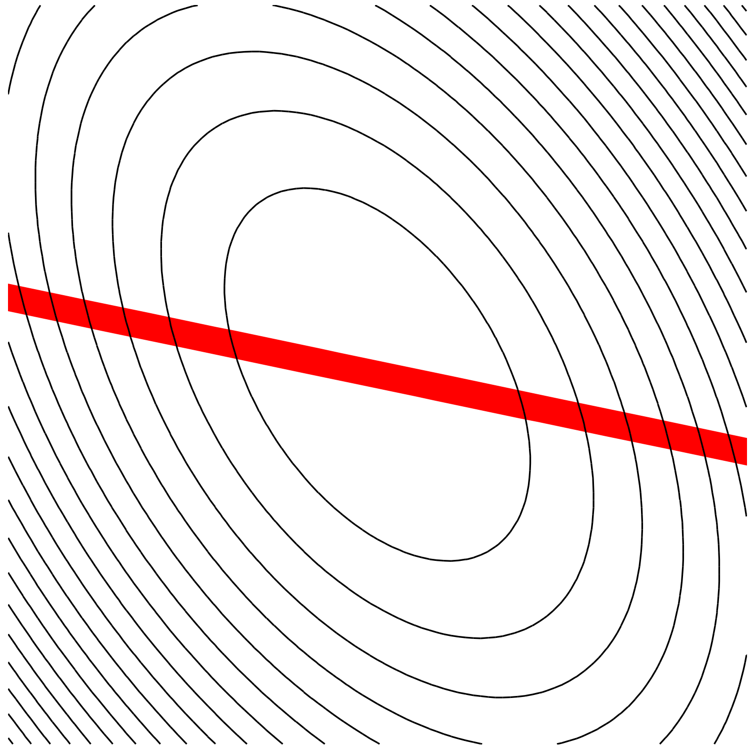


Suggests Lagrangian vortex/eddy/ring with impenetrable boundary.

Eulerian assessment is observer dependent (not objective)

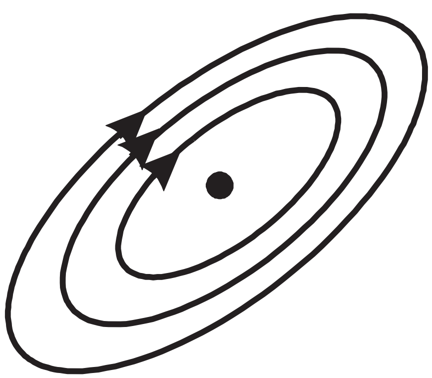


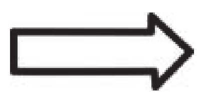
Eulerian assessment is observer dependent (not objective)



$$v(x,t) = \begin{pmatrix} \sin 4t & 2+\cos 4t \\ -2+\cos 4t & -\sin 4t \end{pmatrix} x$$

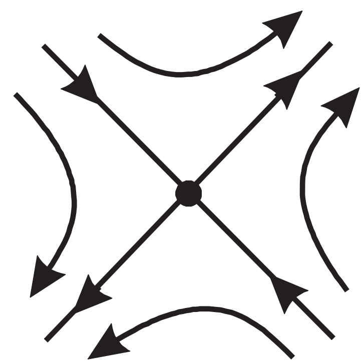
$$\tilde{v}(\tilde{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{x}$$



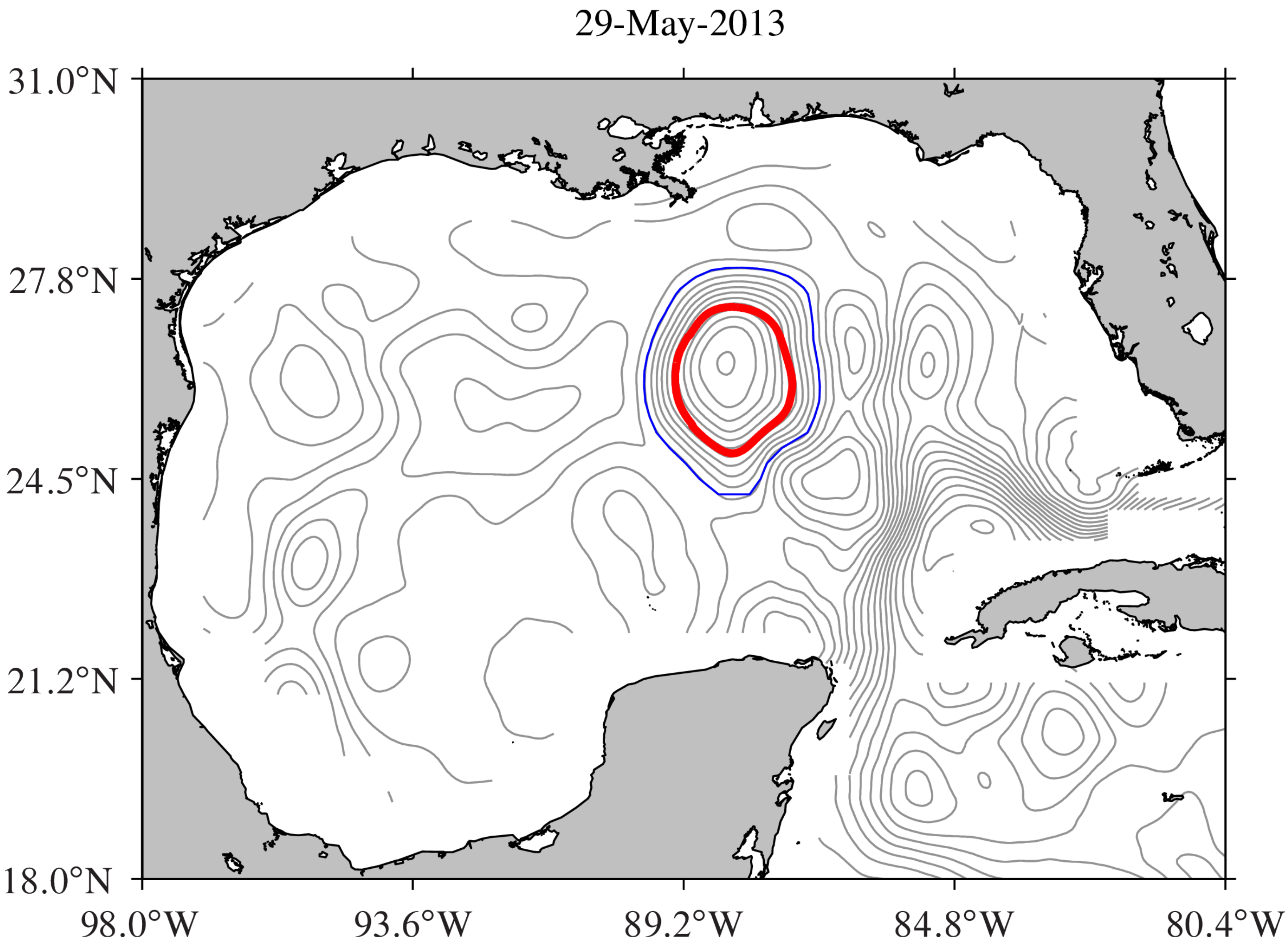


Switch to rotating frame:

$$x = \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix} \tilde{x}$$



Coherent Lagrangian Loop Current ring



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Objective description of material deformation

- Let $v(x,t)$, where $x \in U \subset \mathbb{R}^2$, $t \in [t_0, t_1] \subset \mathbb{R}$ and such that $\nabla \cdot v = 0$, is a (smooth) unsteady incompressible 2-d velocity field.
- Fluid trajectories obey the ***finite-time dynamical system***:

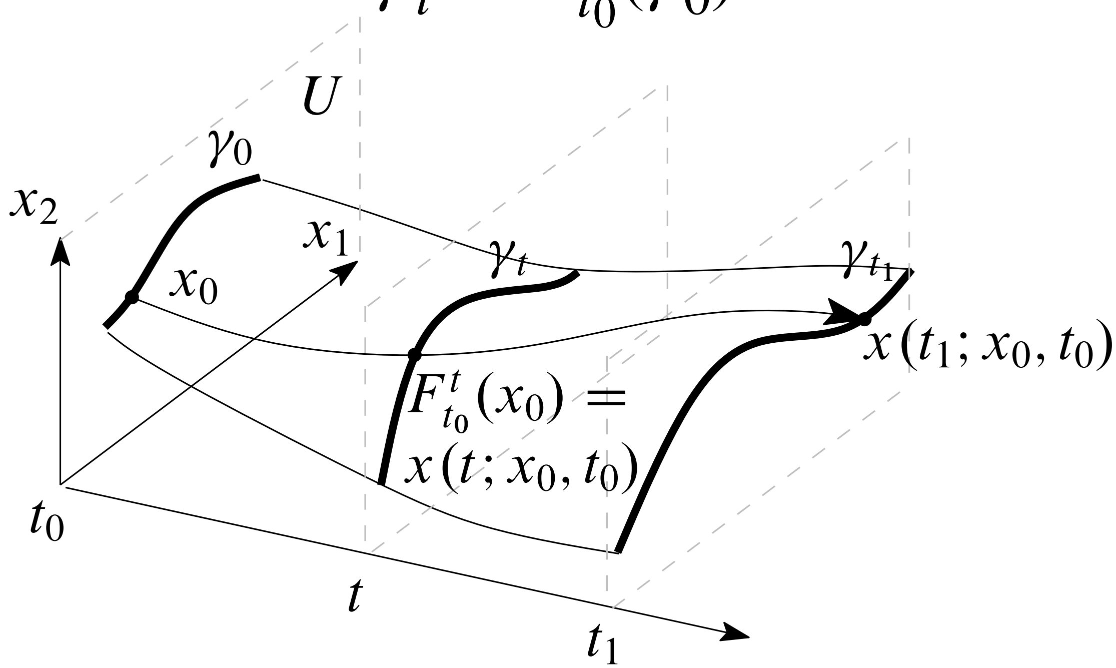
$$\dot{x} = v(x,t). \tag{1}$$

- The ***flow map***,

$$F_{t_0}^{t}(x_0) := x(t; t_0, x_0), \tag{2}$$

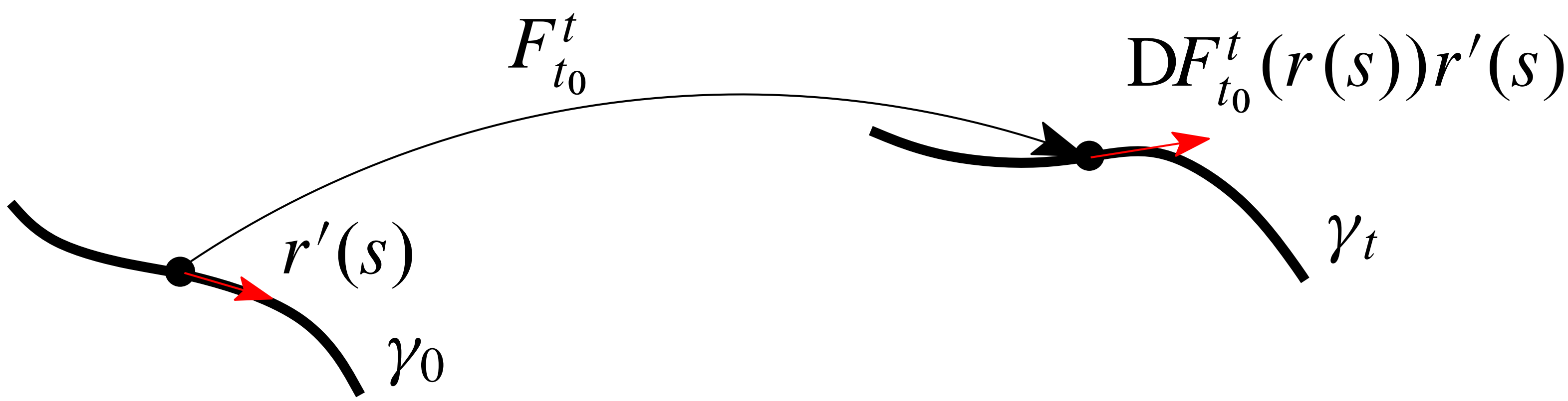
associates time t_0 and t positions.

- Consider a ***material curve*** $\gamma_t = F_{t_0}^{t}(\gamma_0)$.



- Consider the ***local stretching*** of γ_0 under advection.
  - ▷ First select a parameterization for γ_0: $s \mapsto r(s)$.
  - ▷ Tangent to γ_0, $r'(s)$. Tangent vector to γ_t,

$$\frac{\mathrm{d}}{\mathrm{d}s} F_{t_0}^t(r(s)) = \mathrm{D}F_{t_0}^t(r(s))r'(s). \tag{3}$$



  - ▷ The local stretching:

$$\tau(r, r') := \frac{|\mathrm{D}F_{t_0}^t(r)r'|}{|r'|} = \frac{\sqrt{r' \cdot C_{t_0}^t(r)r'}}{\sqrt{r' \cdot r'}}, \tag{4}$$

where

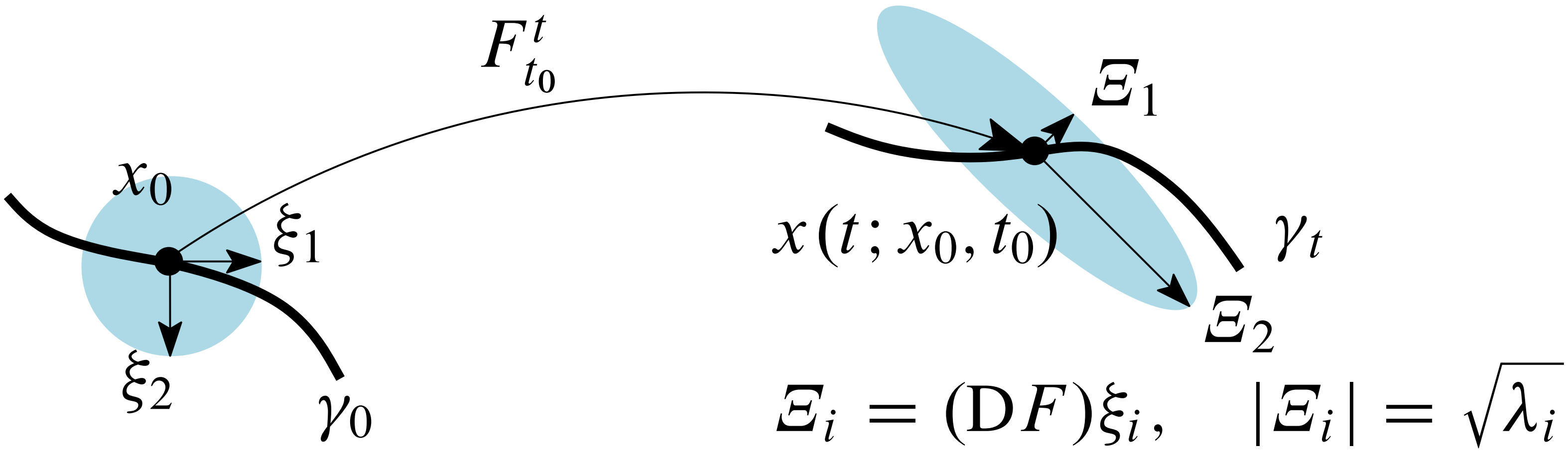
$$C_{t_0}^t := (\mathrm{D}F_{t_0}^t)^\top \mathrm{D}F_{t_0}^t \tag{5}$$

is the (right) ***Cauchy–Green strain tensor***.

- It is important to realize that $\mathrm{D}F$ and C are *objective*.
  - ▷ Consider a change of the observer:
$$x \mapsto \tilde{x} = Q(t)x + b(t), \quad t \mapsto \tilde{t} = t + \alpha, \tag{6}$$
    where $Q \in \mathrm{SO}(2)$ and α is a constant.
  - ▷ A vector u is objective if
$$\tilde{u} = Qu \quad (\Longrightarrow |\tilde{u}| = |u|). \tag{7}$$
  - ▷ A tensor T is objective if
$$\tilde{T} = QTQ^\top \quad (\Longrightarrow \tilde{T}\tilde{u} = QTu). \tag{8}$$
  - ▷ Note that $\mathrm{D}F$ transforms as
$$\tilde{\mathrm{D}}\tilde{F} = \mathrm{D}\tilde{F}Q_0 = \mathrm{D}(QF + b)Q_0^\top = Q\mathrm{D}FQ_0^\top, \tag{9}$$
    so is objective.
  - ▷ Consequently, C transforms as
$$\tilde{C} = (\tilde{\mathrm{D}}\tilde{F})^\top\tilde{\mathrm{D}}\tilde{F} = (Q\mathrm{D}FQ_0^\top)^\top Q\mathrm{D}FQ_0^\top = Q_0CQ_0^\top, \tag{10}$$
    so is also objective.

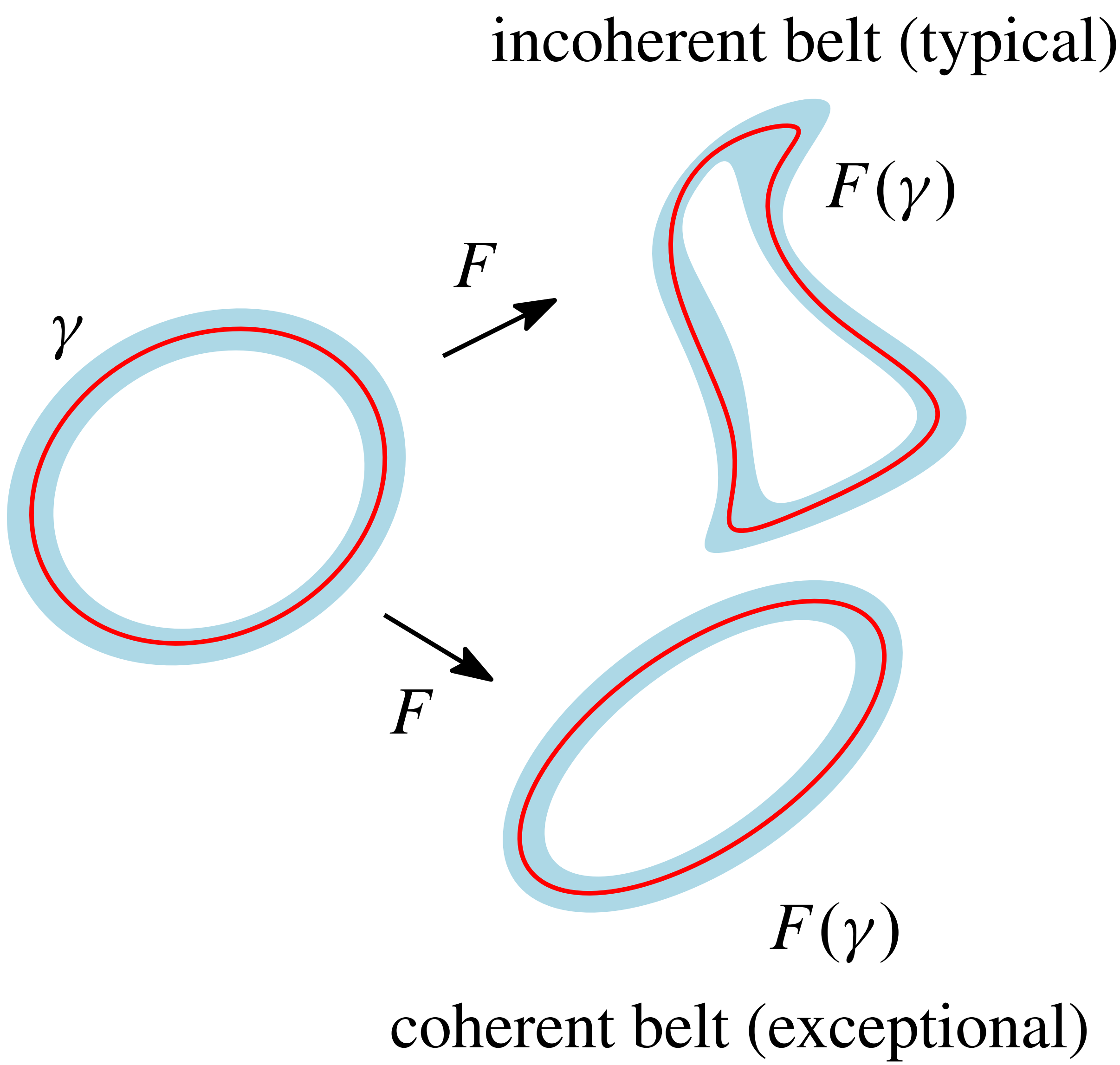
- The CG tensor is *symmetric* and *positive definite* ($u^\top C u = |\mathrm{D}F u|^2 > 0\ \forall u \neq 0$ because $\mathrm{D}F u \neq 0$ as $\mathrm{D}F$ es invertible).
  - ▷ Its eigenvalues $\{\lambda_i(x_0)\}$ and eigenvectors $\{\xi_i(x_0)\}$ satisfy:

$$0 < \lambda_1 \overset{(\nabla\cdot v=0)}{\equiv} \lambda_2^{-1} \leq 1, \quad \xi_i \cdot \xi_j = \delta_{ij}, \quad i = 1, 2. \tag{11}$$



  - ▷ While $\{\xi_i\}$ do *not* have a global orientation, i.e., $C(\pm\xi_i) = \lambda_i(\pm\xi_i)$, tangent curves (called ***tensorlines*** in visualization) are well defined.

Coherent Lagrangian vortex boundaries



Seek material loops with annular neighborhoods showing no leading order change in *averaged stretching*.

Variational problem and solution

- Introduce the functional

$$\mathcal{S}_\tau[r] := \oint \tau(r, r')\,\mathrm{d}s = \oint \frac{\sqrt{r' \cdot C(r) r'}}{\sqrt{r' \cdot r'}}\,\mathrm{d}s, \tag{12}$$

i.e., the stretching integrated along a *loop* $\mathbb{T} \ni s \mapsto r(s) \in \gamma_0$.

- Consider loops $r(s) + \varepsilon h(s)$, $\varepsilon > 0$ small, and require

$$\mathcal{S}_\tau[r + \varepsilon h] = \mathcal{S}_\tau[r] + O(\varepsilon^2) \tag{13}$$

i.e., $\mathcal{S}_\tau$ does not change within thin annular neighborhood of γ_0.

- Same as requiring $\mathcal{S}_\tau$ to be *stationary*:

$$\delta\mathcal{S}_\tau[r](h) = \varepsilon \oint \left(\frac{\partial \tau}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial \tau}{\partial r'} \right) \cdot h\,\mathrm{d}s \equiv 0. \tag{14}$$

- The Euler–Lagrange equation follows:

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial \tau}{\partial r'} - \frac{\partial \tau}{\partial r} = 0, \tag{15}$$

which is an ODE that could be solved, but is 2nd order and does not offer much clarity.

- However, Noether's theorem guarantees:

$$\tau - \frac{\partial \tau}{\partial r'} \cdot r' \equiv \tau = \frac{\sqrt{r' \cdot C(r) r'}}{\sqrt{r' \cdot r'}} = \lambda = \text{const}\,. \tag{16}$$

  ▷ Loops satisfying (15) are *uniformly stretching*—i.e., all their subsets stretch by the same factor λ.

  ▷ Defies universal tendency of material curves to stretch exponentially in turbulence.

- Equation (15) is a 1st-order ODE, but is *implicit*. An *explicit* ODE follows by writing

$$r' = \alpha \xi_1 + \beta \xi_2, \tag{17}$$

which gives:

$$r' = \eta_\lambda^{\pm}(r) := \sqrt{\frac{\lambda_2(r) - \lambda^2}{\lambda_2(r) - \lambda_1(r)}}\, \xi_1(r) \pm \sqrt{\frac{\lambda^2 - \lambda_1(r)}{\lambda_2(r) - \lambda_1(r)}}\, \xi_2(r), \tag{18}$$

whenever $\lambda_1(r) < \lambda^2 < \lambda_2(r)$.

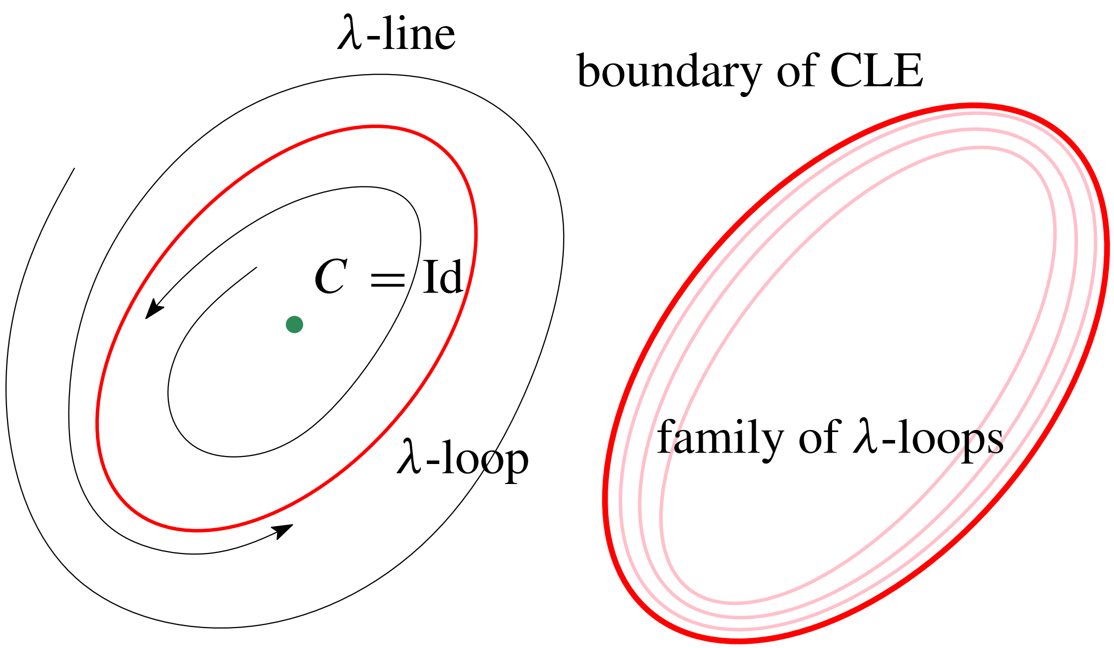
  ▷ Solutions to (17) need not be loops. Closed curves follow as *limit cycles*, which we call ***λ-loops***.

- It can be shown (Karrasch et al. 2015) that a λ-loop must contain at least two points x_0^* such that

$$C(x_0^*) = \mathrm{Id}, \tag{19}$$

which represent ***singularities of C*** where $\lambda_1 = \lambda_2 \overset{(\nabla\cdot v=0)}{=} 1$ and thus $\{\xi_i\}$ are not well defined. Singularities in *line fields* are analogous to critical points in vector fields.

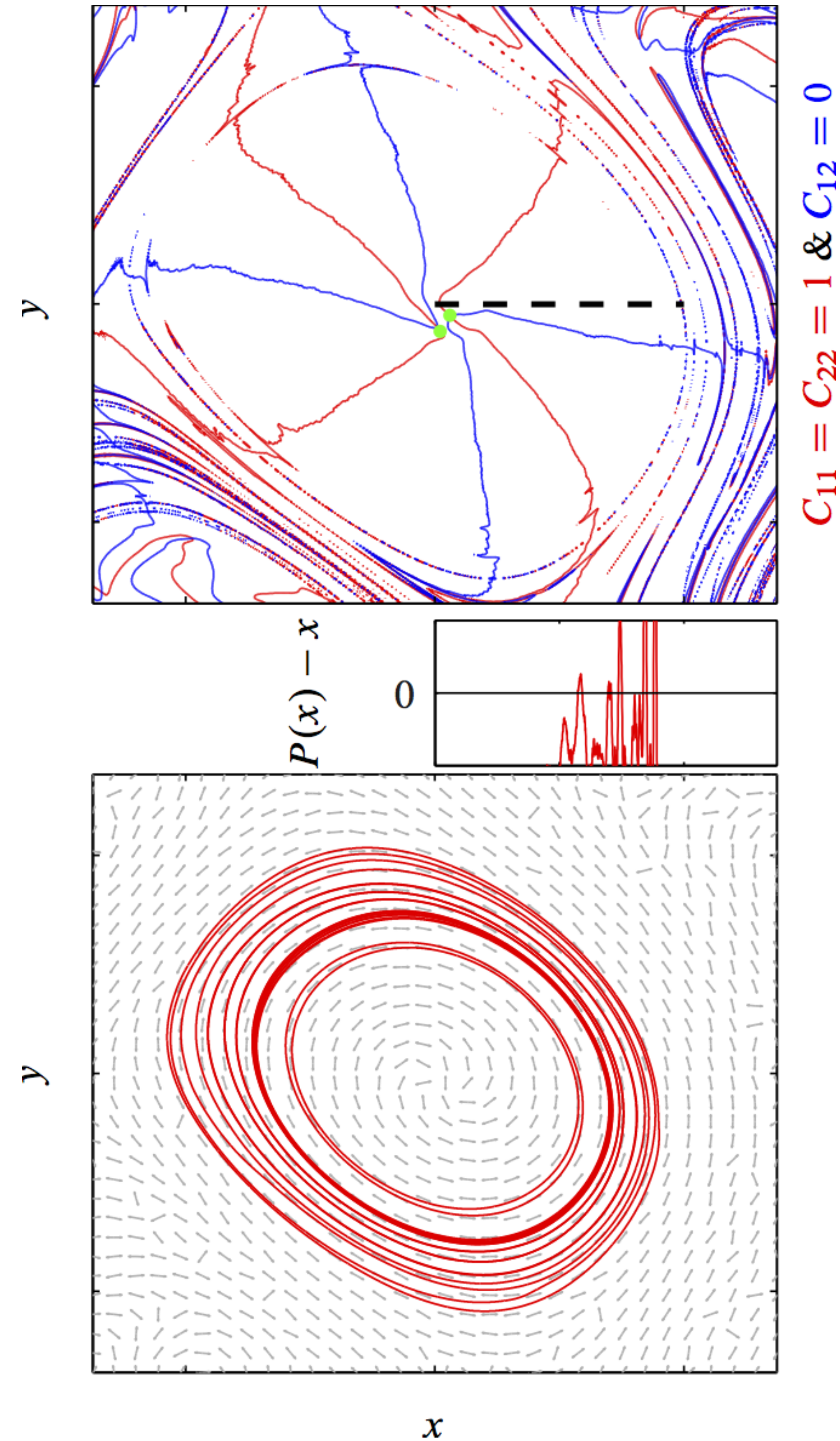
- Furthermore, the λ-loops follow as *nonintersecting* families. The outermost member of a family corresponds to the optimal boundary of a ***coherent Lagrangian eddy***.



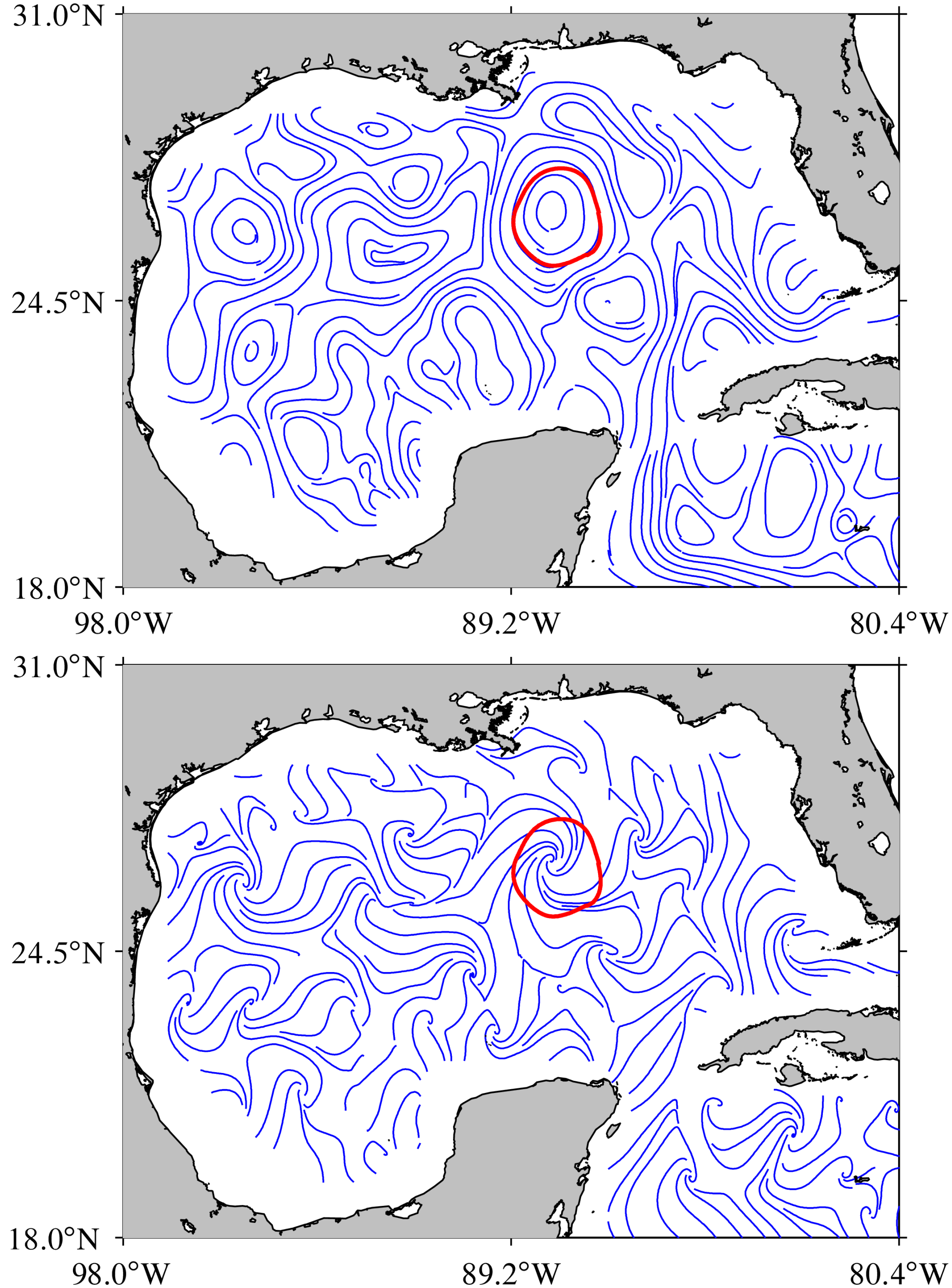
- Typically, the λ-loops exist for $\lambda \approx 1$. As a result, CLE boundaries nearly *reassume their initial arclength at time t*. To see this, let γ_0 be a $\lambda = 1$ loop. Then

$$\begin{aligned}
\operatorname{arclength}(F_{t_0}^{t}(\gamma_0)) &= \oint \sqrt{\eta_1^{\pm}(r) \cdot C_{t_0}^{t}(r)\eta_1^{\pm}(r)}\,\mathrm{d}s \\
&= \oint \sqrt{\frac{1}{1+\lambda_2} + \frac{1}{1+\lambda_1}}\,\mathrm{d}s \\
&= \oint \sqrt{\frac{1}{1+\lambda_2} + \frac{\lambda_2}{1+\lambda_2}}\,\mathrm{d}s \\
&= \operatorname{arclength}(\gamma_0).
\end{aligned} \tag{20}$$

- *Enclosed area conservation* further conveys Lagrangian vortices exceptional coherence.
- Finally, the λ-loops are *robust*: they survive under flow perturbations (limit cycles are *structurally stable*). Consequential for dealing with *imperfect* data.

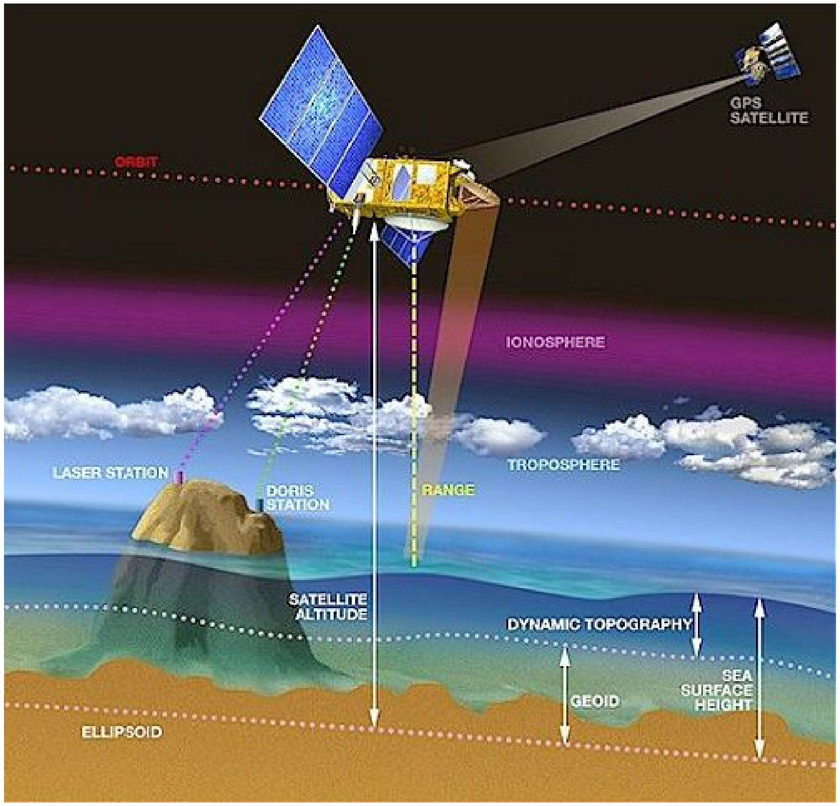


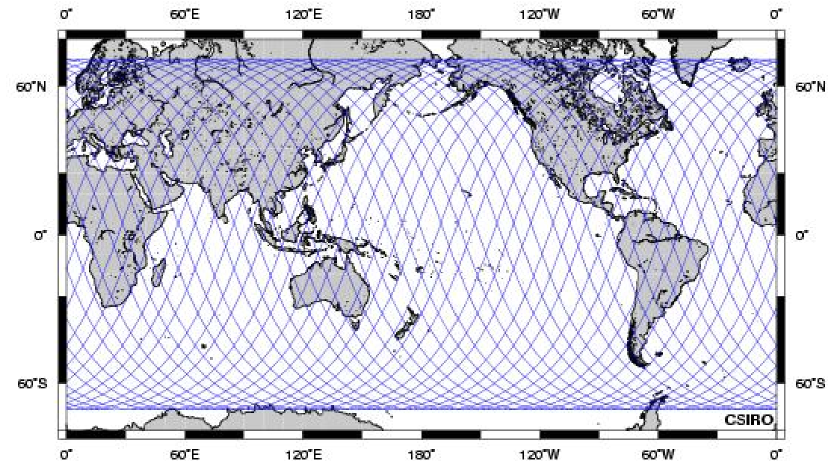
$C_{11} = C_{22} = 1$ & $C_{12} = 0$
y
$P(x) - x$
0
y
x



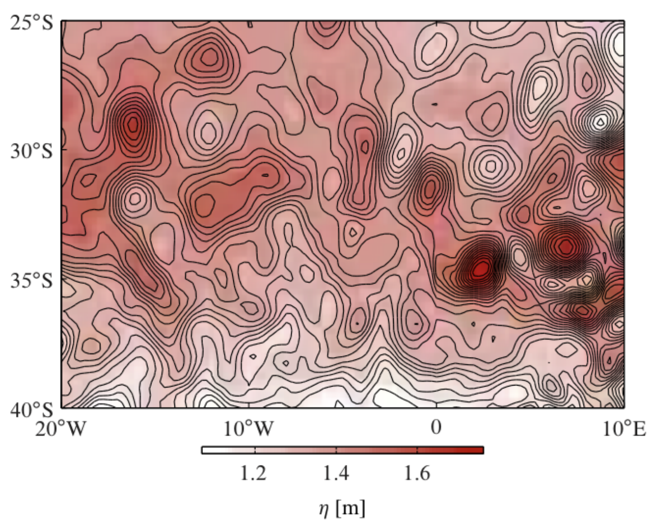
Obtained from $t - t_0 = 200$ d integration with $\lambda = 1.05$.

Velocity derived from satellite altimetry



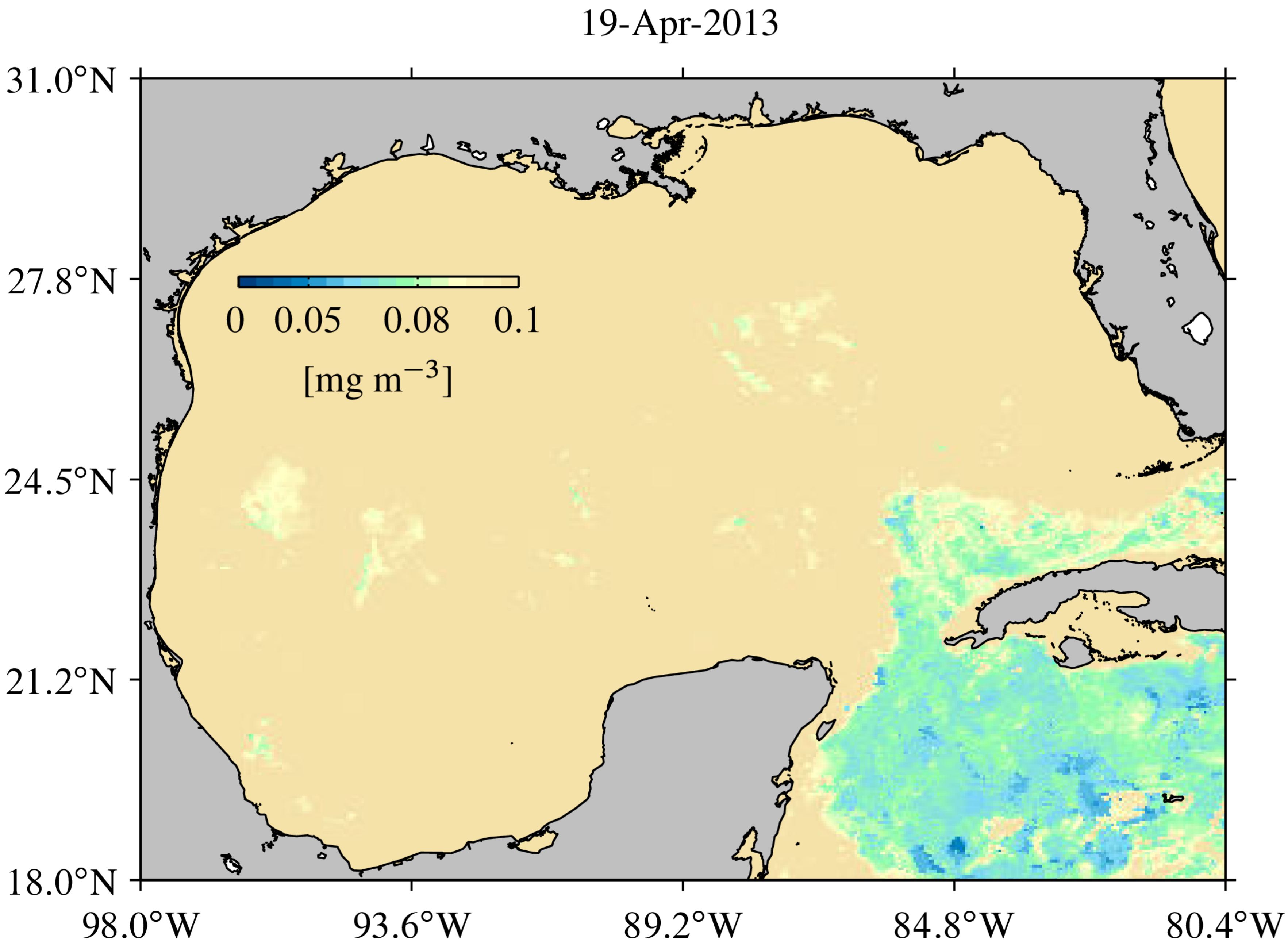


The ground track of the first three days of the 10-day cycle of TOPEX/Poseidon

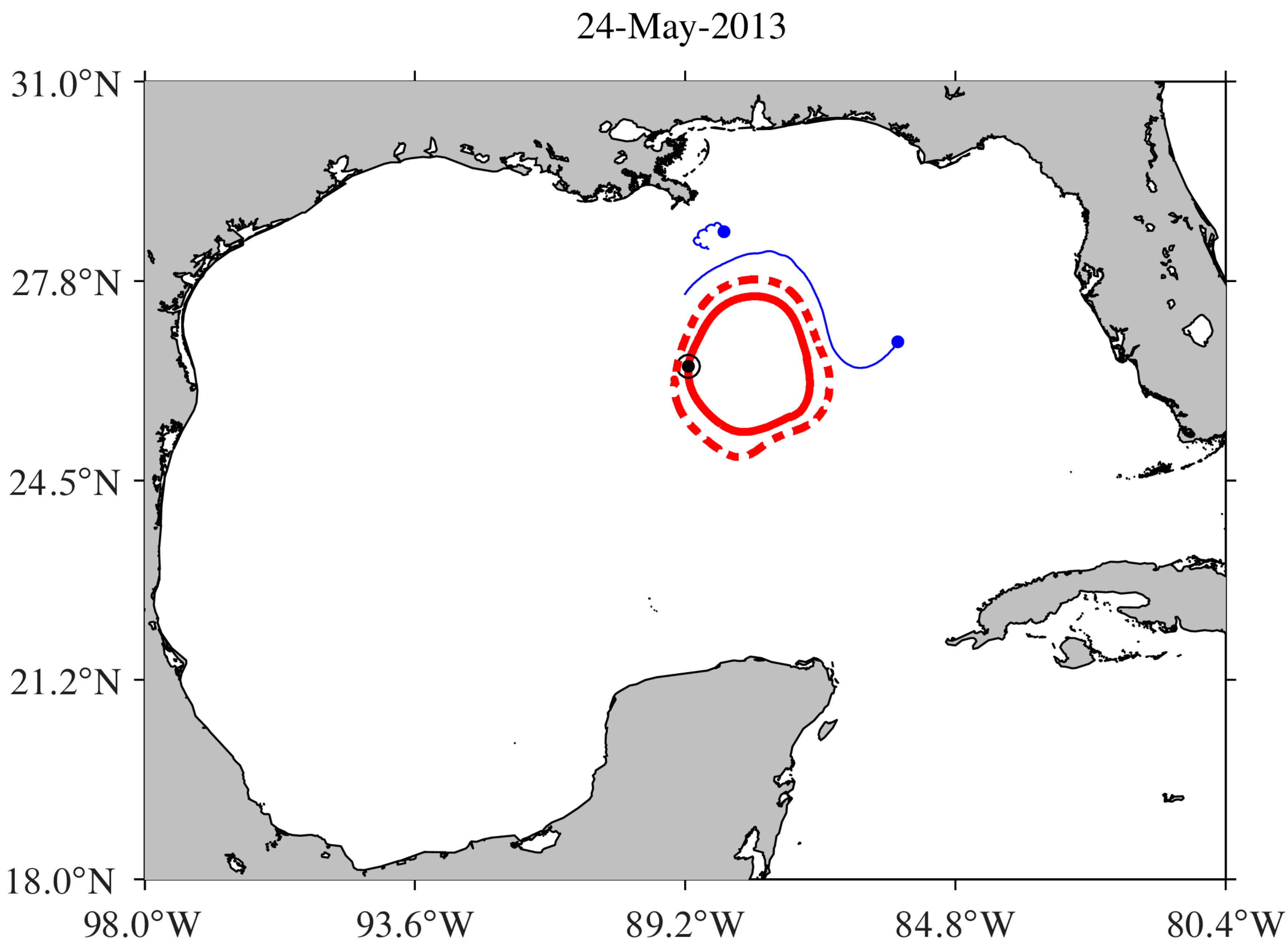


- Altimeter measures SSH anomaly.
- Global maps since 1992.
- Weekly data, ∼25 km.
- Distributed by AVISO.
- Assumption: $v = g\nabla^{\perp}\eta/f$.

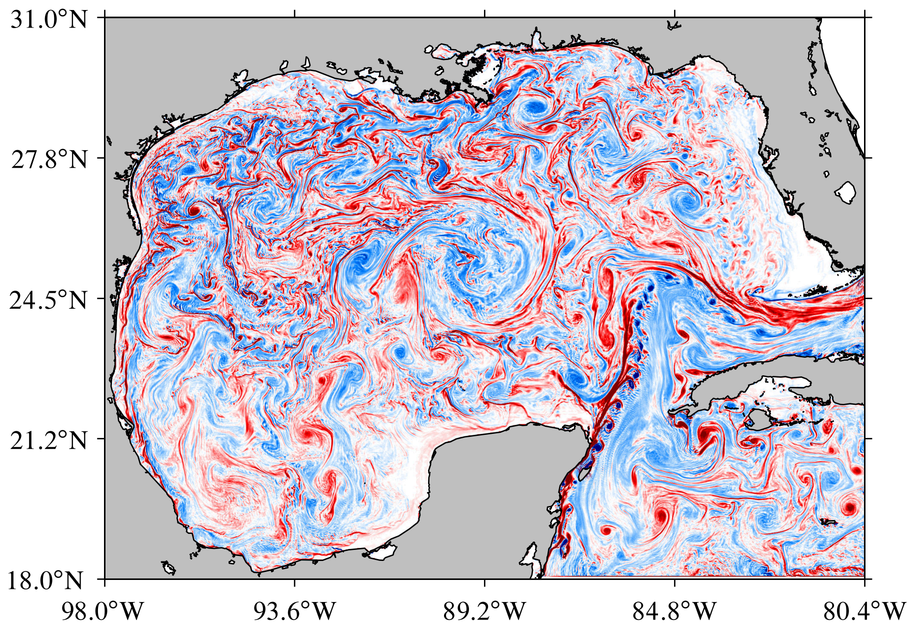
Satellite-derived chlorophyll

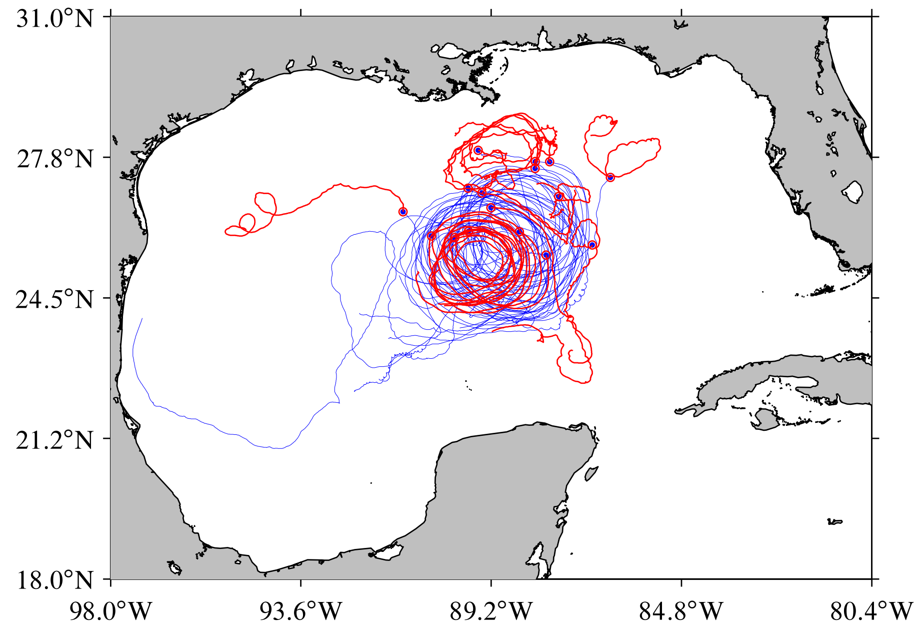


Satellite-tracked drifters

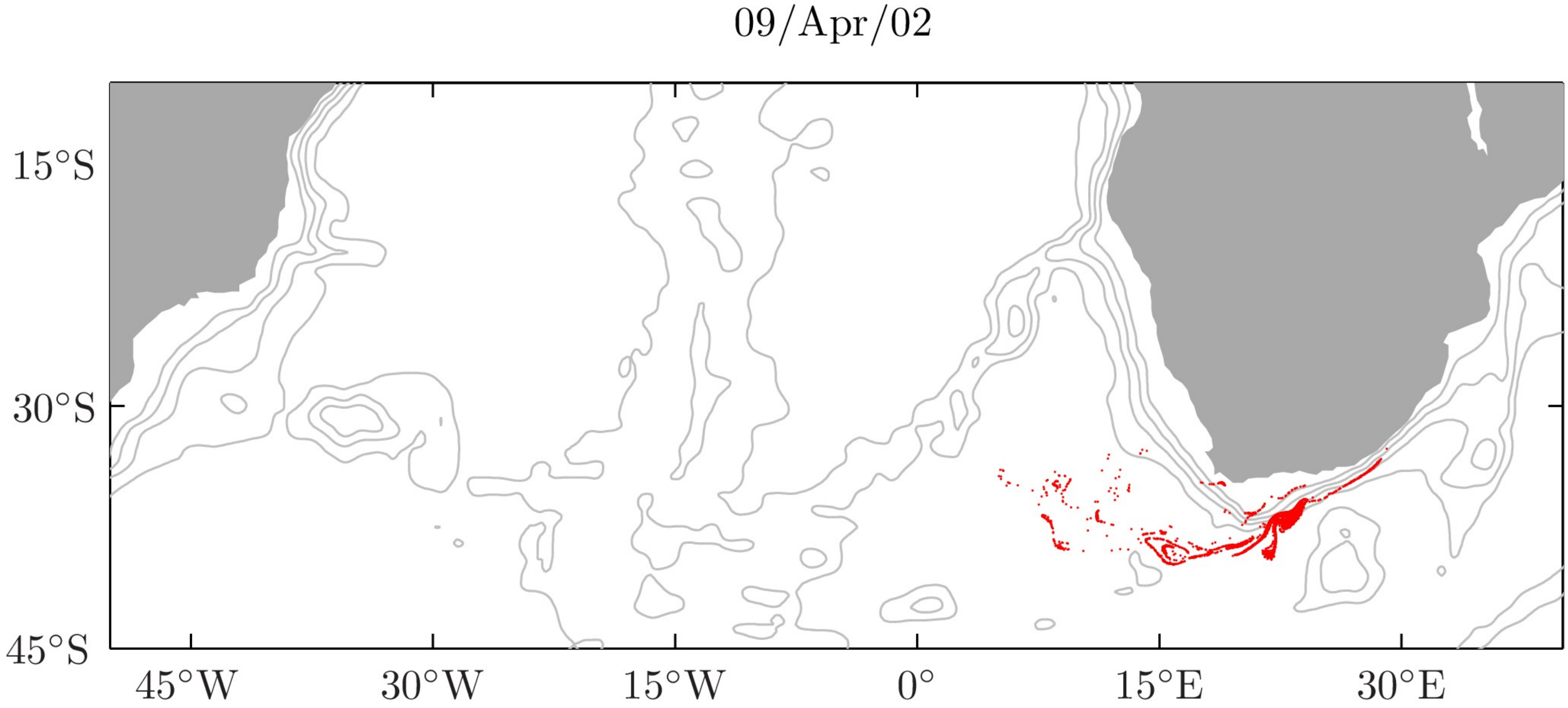


Compare with submesoscale-permitting simulation

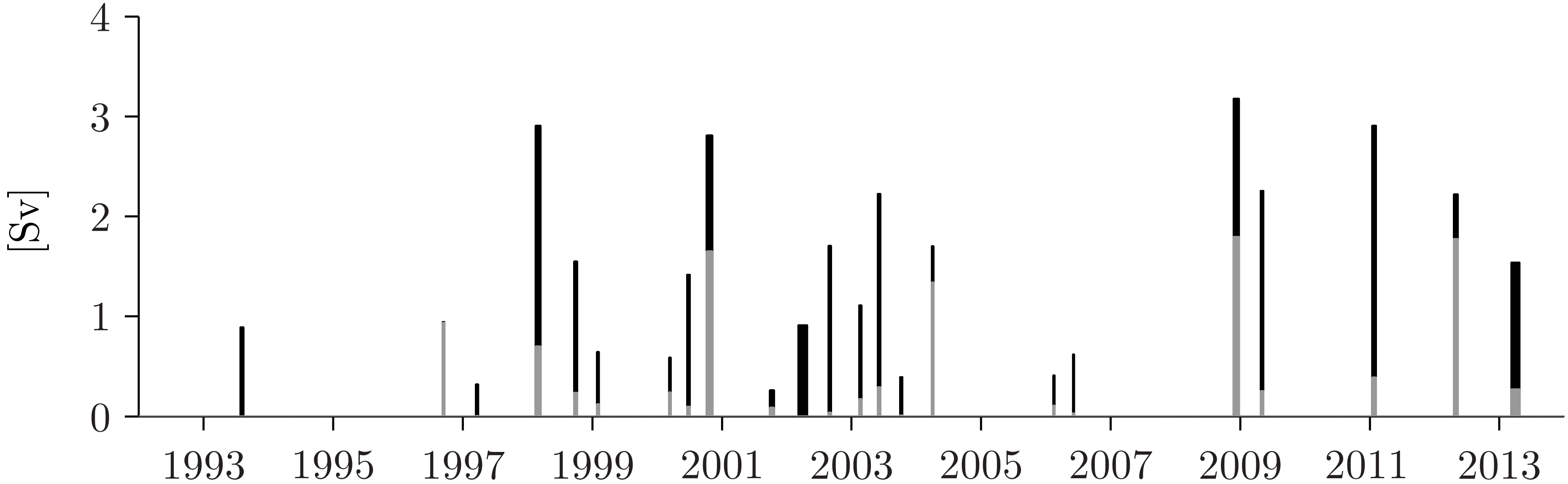


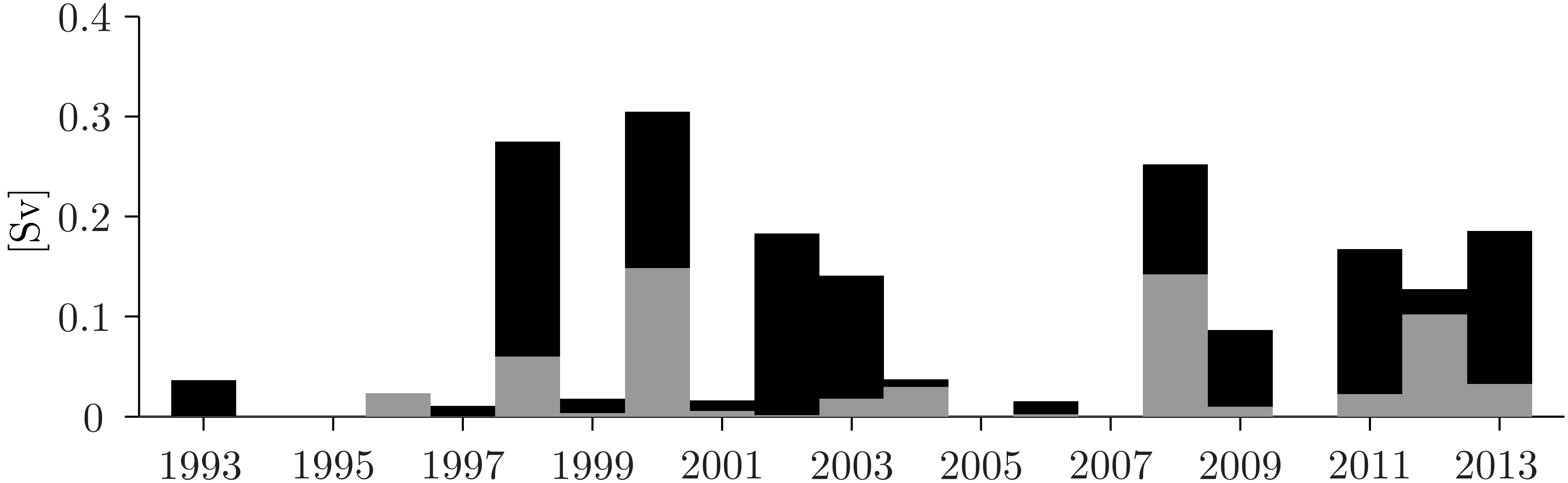


Possible climatic consequences

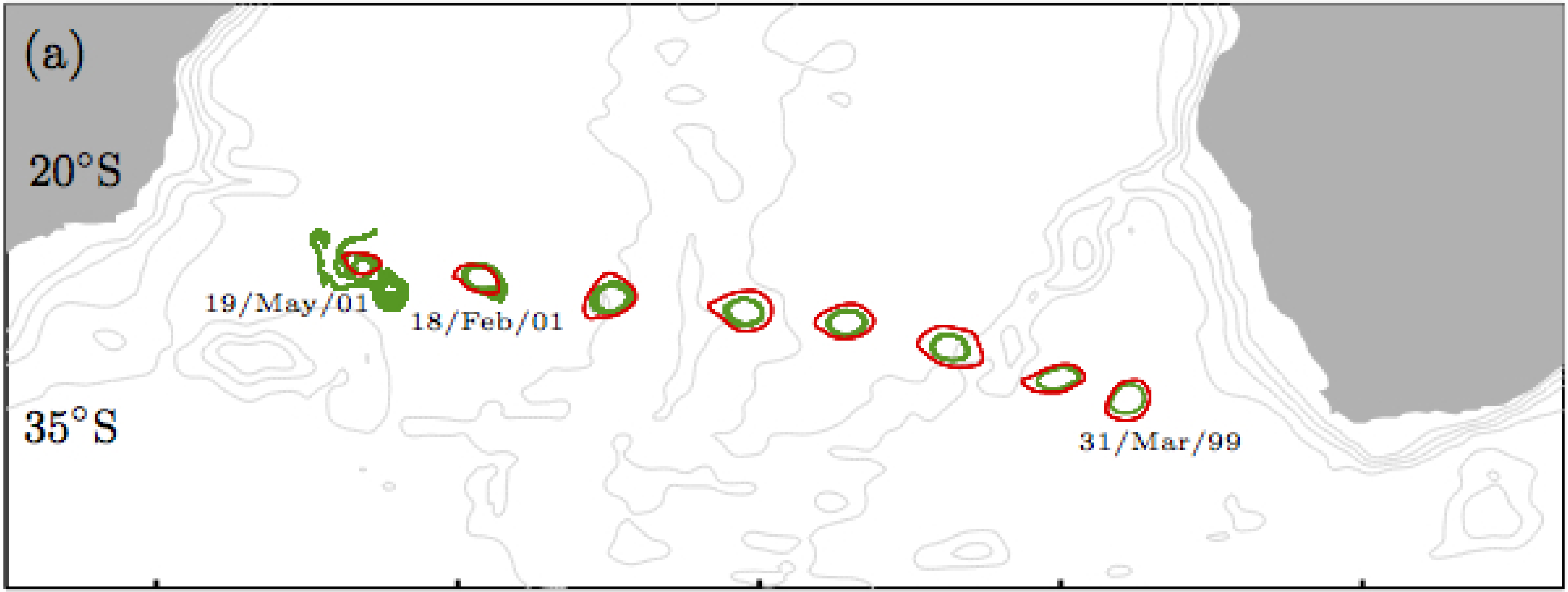


Coherent transport





Possibly larger due to coherence re-gaining



Final remarks

- CLE detection now coded in *Lagrangian Coherent Structure* (LCS) Toolkit (Onu et al., 2015).
- The λ-loops are geodesics—can be seen through appropriate reparametrization that makes Lagrangian quadratic. The metric is Lorentzian and vanishes so geodesics are null—leads to cosmological analogy.
- The λ-loops are generalized KAM tori (i.e., elliptic LCS)—approximate KAM tori with few iterations. Can KAM theory explain the occurrence of CLE? Ongoing work with R. de la Llave (Georgia Tech).
- CLE can be extracted levelwise to draw a 3-d picture—most appropriate perhaps in mesoscale flows. Full 3-d analysis is possible using an objective vorticity-based approach (Haller et al., 2016).
- Work in progress is the characterization of swirling structures in dissipative velocity fields.

Escuela interdisciplinaria de transporte en fluidos geofísicos: de los remolinos oceánicos a los agujeros negros

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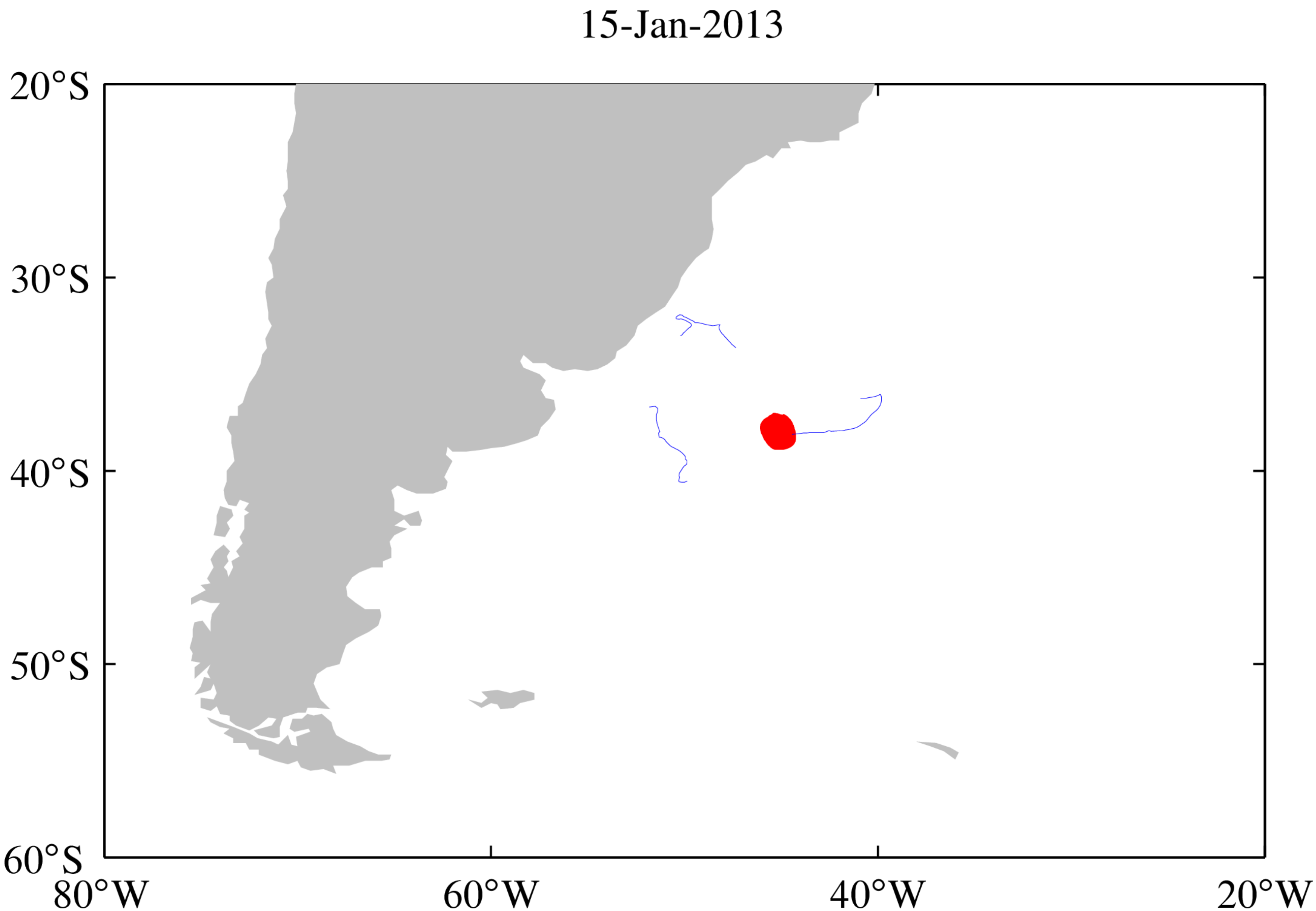
Del 5 al 16 de diciember 2016.

PhD thesis theme in RSMAS?



Thank you.